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Evaluation of Effective Resistances in Pseudo-Distance-Regular Resistor Networks

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Abstract The effective resistance or two-point resistance between two nodes of a resistor network is the potential difference that appears across them when a unit current source is applied between the nodes as terminals. This concept arises in problems which deal with graphs as electrical networks including random walks, distributed detection and estimation, sensor networks, distributed clock synchronization, collaborative filtering, clustering algorithms and etc. In the previous paper (Jafarizadeh et al. in J. Math. Phys. 50:023302, 2009) a recursive formula for evaluation of effective resistances on the so-called distance-regular networks was given based on the Christoffel-Darboux identity. In this paper, we consider more general networks called pseudo-distance-regular networks or *QD* type networks, where we use the stratification of these networks and show that the effective resistances between a given node, say α, and all of the nodes β belonging to the same stratum with respect to α, are the same. Then, based on the spectral techniques, for those α, β's which satisfy $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ (L^{-1} is the pseudo-inverse of the Laplacian of the network), an analytical formula for effective resistances $R_{\alpha\beta^{(m)}}$ (the equivalent resistance between terminals α and β, so that β belongs to the m-th stratum with respect to α) is given in terms of the first and second orthogonal polynomials associated with the network. From the fact that in distance-regular networks, $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ is satisfied for all nodes α, β of the network, the effective resistances

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$R_{\alpha\beta^{(m)}}$ for $m = 1, 2, \ldots, d$ (d is diameter of the network which is the same as the number of strata) are calculated directly, by using the given formula.

Keywords Effective resistance · Pseudo-distance-regular networks · Stratification · Spectral distribution

1 Introduction

The study of electric networks was formulated by Kirchhoff [2] more than 150 years ago as an instance of a linear analysis. Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [3]), the theory of harmonic functions [4] to lattice Green's functions [5, 6]. As it is well-known, the effective resistance (two-point resistance) between two nodes of a resistor network is the potential difference that appears across them when a unit current source is applied between the nodes as terminals. The effective resistance and molecular structure descriptors based on it were much studied in the chemical literature [7, 8]. It is also closely related to average first passage time and average commute time which are two important quantities in random walk models defined based on Markov chains. It is shown in Ref. [9] that computation of average commute time can be obtained via the Moore-Penrose generalized inverse of the (combinatorial) Laplacian matrix $L = D - A$, where A is the adjacency matrix of the underlying network (graph) and D is the diagonal matrix in which the ith diagonal entry is d_i (the degree of vertex i). Also, it has been shown that this quantity and its square root are distance, since it can be shown that L^{-1} is symmetric and positive semidefinite. It is therefore called the Euclidean commute time (ECT) distance. In fact, ECT distance is the same as effective resistance (effective resistance is symmetric and satisfies the triangle inequality and so is a distance metric and sometimes is called resistance distance). Therefore, any clustering algorithm (hierarchical clustering, k-means, etc.) [10], which can be used in conjunction with ECT distance, deals with effective resistance. In fact, the concept of resistance distance between two vertices, introduced by Klein and Randic [11] (defined to be the effective resistance between the two vertices, when the graph is viewed as an electrical network with each edge carrying unit resistance) in order to examine other possible metrics in (molecular) graphs, is intrinsic to the graph with some nice purely mathematical interpretations and other interpretations [7, 12]. On the other hand, there is the long recognized shortest path distance function which has been extensively studied and found many applications [13, 14]. For these two distance functions, the shortest-path might be imagined to be more relevant when there is corpuscular communication (along edges) between two vertices, whereas the resistance distance might be imagined to be more relevant when the communication is wave-like. That the communication of many things is rather wave-like, such as, chemical communication in molecules and information communication in networks, suggests a substantial potential for applications, beyond the traditional electrical ones. Moreover, resistor network modeling is relevant to problems including distributed detection and estimation [15, 16], sensor networks [17], distributed clock synchronization [18], collaborative filtering, routing algorithms, clustering algorithms [19] and etc. The connection between these problems and electrical networks originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. In the problem of estimating vector valued variables from noisy "relative" measurements, the measurement model can be expressed in terms of a graph, whose nodes correspond to the variables being estimated and the edges to noisy measurements of the difference between the two variables associated with the corresponding nodes (i.e., their relative values). This type of measurement model appears in several applications such as sensor network localization, time synchronization, and motion consensus. In [20], a characterization on the minimum possible covariance of the estimation error has been obtained, when an arbitrarily large number of measurements are available. This covariance has been shown to be equal to a matrix-valued effective resistance in an infinite electrical network. Covariance in large finite graphs converges to this effective resistance as the size of the graphs increases.

Recently, the authors have given a method for calculation of the effective resistance on distance-regular networks [21], where the calculation is based on stratification introduced in [22] and Stieltjes transform of the spectral distribution (Stieltjes function) associated with the network. Also, in [21] it has been shown that the resistances between a node α and all nodes β belonging to the same stratum with respect to the α ($R_{\alpha\beta^{(i)}}$, β belonging to the i-th stratum with respect to the α) are the same and the analytical formulas have been given for two-point resistances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, 3$ in terms of the size of the network and corresponding intersection array without any need to know the spectrum of the pseudo inverse L^{-1}. In the next work [1], the authors have used the algebraic structure of distance-regular networks (Bose-Mesner algebra) such as stratification and spectral techniques specially the well known Christoffel-Darboux identity [23] from the theory of orthogonal polynomials to give a recursive formula for calculation of all of the resistance distances $R_{\alpha\beta^{(i)}}$, $i = 1, 2, \ldots, d$ (d is diameter of the network which is the same as the number of strata) on the network without any need to calculating the spectrum of the pseudo inverse of the Laplacian of the network denoted by L^{-1}. In this way they have shown that, the effective resistance strictly increases by increasing the shortest path distance defined on the network, i.e., $R_{\alpha\beta^{(m+1)}} - R_{\alpha\beta^{(m)}} > 0$ for all $m = 1, 2, \ldots, d-1$. Here in this work, evaluation of effective resistances on more general networks called pseudo-distance-regular networks [24] or QD type networks [25] is investigated, where we use the stratification of these networks and show that the effective resistances between a given node such as α and all of the nodes β belonging to the same stratum with respect to α are the same. Then, based on the spectral techniques, an analytical formula for effective resistances $R_{\alpha\beta^{(m)}}$ such that $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ (those nodes α, β of the network such that the network is symmetric with respect to them) is given in terms of the first and second orthogonal polynomials associated with the network. Particularly, due to the fact that, in distance-regular networks we have $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ for all nodes α, β of the network, the effective resistances $R_{\alpha\beta^{(m)}}$ for $m = 1, 2, \ldots, d$ are calculated directly, by using the given formula.

The organization of the paper is as follows. In Sect. 2, we give some preliminaries such as definitions related to electrical effective resistance, graphs, their adjacency matrices, stratification of the graphs, pseudo-distance-regular graphs and spectral distribution associated with the graphs. In Sect. 3, an analytical formula for calculating the effective resistances in pseudo-distance-regular graphs as resistor networks is given in terms of the orthogonal polynomials of the first kind and second kind associated with the networks. The paper is ended with a brief conclusion and an Appendix.

2 Preliminaries

In this section we give some preliminaries such as definitions related to electrical effective resistance, graphs, corresponding stratification, pseudo-distance-regular graphs and spectral distribution techniques.

2.1 Effective Resistances in Resistor Networks

A classic problem in electric circuit theory studied by numerous authors over many years, is the computation of the resistance between two nodes in a resistor network (see, e.g., [26]).

A resistor network can be considered as an undirected graph which is a set of objects called vertices or nodes, where some unordered pairs (but sets $\{x, y\}$) of the vertices are connected by links called edges. Let $\Gamma = (V, E)$ be such a graph with $v = |V|$ nodes and $u = |E|$ edges. Then, Γ can be viewed as a resistor network, where $r_{ij} = r_{ji}$ is considered as the resistance of the resistor connecting nodes i and j. Hence, the conductance is $c_{ij} = r_{ij}^{-1} = c_{ji}$ so that $c_{ij} = 0$ if there is no resistor connecting i and j (the conductance c_{ij} is considered as the weight on edge connecting nodes i and j). The effective resistance between a pair of nodes i and j, denoted by R_{ij}, is the electrical resistance measured across nodes i and j, when the network represents an electrical circuit with each edge (or branch, in the terminology of electrical circuit) a resistor with (electrical) conductance c_{ij}. In other words, R_{ij} is the potential difference that appears across terminals i and j when a unit current source is applied between them. It can be noted that, R_{ij} is a measure of how 'close' the nodes i and j are: R_{ij} is small when there are many paths between nodes i and j with high conductance edges, and R_{ij} is large when there are few paths, with lower conductance, between nodes i and j. Indeed, the effective resistance R_{ij} is sometimes referred to as the resistance distance between nodes i and j.

Denote the electric potential at the i-th vertex by V_i and the net current flowing into the network at the i-th vertex by I_i (which is zero if the i-th vertex is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint $\sum_{i=1}^{v} I_i = 0$. The Kirchhoff law states

$$\sum_{j=1, j\neq i}^{v} c_{ij}(V_i - V_j) = I_i, \quad i = 1, 2, \dots, v. \tag{2.1}$$

Explicitly, (2.1) reads

$$L\vec{V} = \vec{I}, \tag{2.2}$$

where, $\vec{V}$ and $\vec{I}$ are v-vectors whose components are V_i and I_i, respectively and

$$L = \sum_i c_i |i\rangle\langle i| - \sum_{i,j} c_{ij} |i\rangle\langle j| \tag{2.3}$$

is the Laplacian of the graph Γ with

$$c_i \equiv \sum_{j=1, j\neq i}^{v} c_{ij}, \tag{2.4}$$

for each vertex α. It should be noticed that, L has eigenvector $(1, 1, \dots, 1)^t$ with eigenvalue 0. Therefore, L is not invertible and so we define the pseudo-inverse of L as

$$L^{-1} = \sum_{i, \lambda_i \neq 0} \lambda_i^{-1} E_i, \tag{2.5}$$

where, E_i is the operator of projection onto the eigenspace of L^{-1} corresponding to eigenvalue λ_i. It has been shown that, the two-point resistances $R_{\alpha\beta}$ are given by

$$R_{\alpha\beta} = \langle\alpha|L^{-1}|\alpha\rangle + \langle\beta|L^{-1}|\beta\rangle - \langle\alpha|L^{-1}|\beta\rangle - \langle\beta|L^{-1}|\alpha\rangle. \tag{2.6}$$

This formula may be formally derived using Kirchoff 's laws, and seems to have been long known in the electrical engineering literature, with its appearing in several texts, such as [27].

2.2 Graph and Its Adjacency Matrix

As defined in the previous subsection, a graph is a pair $\Gamma = (V, E)$, where V is a non-empty set and E is a subset of $\{(x, y) : x, y \in V, x \neq y\}$. Elements of V and of E are called vertices and edges, respectively. Two vertices $x, y \in V$ are called adjacent if $(x, y) \in E$, and in that case we write $x \sim y$. For a graph $\Gamma = (V, E)$, the adjacency matrix A is defined as

$$\big(A\big)_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha \sim \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Conversely, for a non-empty set V, a graph structure is uniquely determined by such a matrix indexed by V. The degree or valency of a vertex $x \in V$ is defined by

$$\deg(x) \equiv \kappa(x) = |\{y \in V : y \sim x\}| \tag{2.8}$$

where, $|\cdot|$ denotes the cardinality. The graph is called regular if the degree of all of the vertices be the same. In this paper, we will assume that graphs under discussion are regular. A finite sequence $x_0, x_1, \ldots, x_n \in V$ is called a walk of length n (or of n steps) if $x_{i-1} \sim x_i$ for all $i = 1, 2, \ldots, n$. Let $l^2(V)$ denote the Hilbert space of C-valued square-summable functions on V. With each $\beta \in V$ we associate a vector $|\beta\rangle$ such that the β-th entry of it is 1 and all of the other entries of it are zero. Then $\{|\beta\rangle : \beta \in V\}$ becomes a complete orthonormal basis of $l^2(V)$. The adjacency matrix is considered as an operator acting in $l^2(V)$ in such a way that

$$A|\beta\rangle = \sum_{\alpha\sim\beta} |\alpha\rangle. \tag{2.9}$$

2.3 Stratification

For $x \neq y$ let $\partial(x, y)$ be the length of the shortest walk connecting x and y. By definition $\partial(x, x) = 0$ for all $x \in V$. The graph becomes a metric space with the distance function ∂. Note that $\partial(x, y) = 1$ if and only if $x \sim y$. We fix a point $o \in V$ as an origin of the graph, called reference vertex. Then, the graph Γ is stratified into a disjoint union of strata:

$$V = \bigcup_{i=0}^{\infty} \Gamma_i(o), \qquad \Gamma_i(o) := \{\alpha \in V : \partial(\alpha, o) = i\}. \tag{2.10}$$

Note that $\Gamma_i(o) = \emptyset$ may occur for some $i \geq 1$. In that case we have $\Gamma_i(o) = \Gamma_{i+1}(o) = \cdots = \emptyset$. With each stratum $\Gamma_i(o)$ we associate a unit vector in $l^2(V)$ defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha\in\Gamma_i(o)} |\alpha\rangle, \tag{2.11}$$

where, $\kappa_i = |\Gamma_i(o)|$ is called the i-th valency of the graph ($\kappa_i := |\{\gamma : \partial(o, \gamma) = i\}| = |\Gamma_i(o)|$). The closed subspace of $l^2(V)$ spanned by $\{|\phi_i\rangle\}$ is denoted by $\Lambda(\Gamma)$. Since $\{|\phi_i\rangle\}$ becomes a complete orthonormal basis of $\Lambda(\Gamma)$, we often write

$$\Lambda(\Gamma) = \sum_i \oplus \mathbf{C}|\phi_i\rangle. \tag{2.12}$$

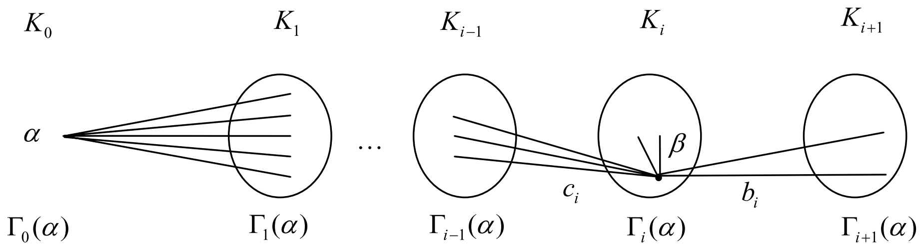

Fig. 1 Shows edges through α and β in a pseudo-distance-regular graph

2.4 Pseudo-Distance-Regular Graphs

Given a vertex $\alpha \in V$ of a graph Γ, consider the stratification (2.10) with respect to α such that $\Gamma_i(\alpha) = \emptyset$ for $i > d$. Then, we say that Γ is pseudo-distance-regular around vertex α whenever for any $\beta \in \Gamma_k(\alpha)$ and $0 \le k \le d$, the numbers

$$c_k(\beta) := \frac{1}{\kappa(\beta)} \sum_{\gamma\in\Gamma_1(\beta)\cap\Gamma_{k-1}(\alpha)} \kappa(\gamma), \qquad a_k(\beta) := \frac{1}{\kappa(\beta)} \sum_{\gamma\in\Gamma_1(\beta)\cap\Gamma_{k}(\alpha)} \kappa(\gamma),$$
$$b_k(\beta) := \frac{1}{\kappa(\beta)} \sum_{\gamma\in\Gamma_1(\beta)\cap\Gamma_{k+1}(\alpha)} \kappa(\gamma), \tag{2.13}$$

do not depend on the considered vertex $\beta \in \Gamma_k(\alpha)$, but only on the value of k. In such a case, we denote them by c_k, a_k and b_k respectively. Then, the matrix

$$\begin{pmatrix} 0 & c_1 & \dots & c_{d-1} & c_d \\ a_0 & a_1 & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_{d-1} & 0 \end{pmatrix} \tag{2.14}$$

is called the (pseudo-)intersection array around vertex α of Γ. It is shown in [24] that this is a generalization of the concept of distance-regularity around a vertex (which in turn is a generalization of distance-regularity). It should be noticed that for regular graphs Γ ($\kappa(\beta) = \kappa$ for all $\beta \in V$), the numbers c_k, a_k and b_k read as

$$c_k = |\Gamma_1(\beta) \cap \Gamma_{k-1}(\alpha)|, \qquad a_k = |\Gamma_1(\beta) \cap \Gamma_k(\alpha)|, \qquad b_k = |\Gamma_1(\beta) \cap \Gamma_{k+1}(\alpha)|, \tag{2.15}$$

where we tacitly understand that $\Gamma_{-1}(\alpha) = \emptyset$ (see Fig. 1). The intersection numbers (2.15) and the valencies $\kappa_i = |\Gamma_i(\alpha)|$ satisfy the following obvious conditions

$$a_i + b_i + c_i = \kappa, \qquad \kappa_{i-1}b_{i-1} = \kappa_i c_i, \quad i = 1, \dots, d,$$
$$\kappa_0 = c_1 = 1, \qquad b_0 = \kappa_1 = \kappa, \qquad (c_0 = b_d = 0). \tag{2.16}$$

One should notice that, the definition of pseudo-distance regular graphs together with (2.16), imply that in general, the valencies κ_i (the size of the i-th stratum) for $i = 0, 1, \dots, d$ do not depend on the considered vertex $\beta \in \Gamma_k(\alpha)$, but only on the value of k.

The notion of pseudo-distance regularity has a close relation with the concept of QD type graphs introduced by Obata [25], such that for the adjacency matrices of this type of graphs, one can obtain a quantum decomposition associated with the stratification (2.10) as

$$A = A^+ + A^- + A^0, \tag{2.17}$$

where, the matrices A^+, A^- and A^0 are defined as follows: for $\beta \in \Gamma_k(\alpha)$, we set

$$(A^+)_{\beta\delta} = \begin{cases} A_{\beta\delta} & \text{if } \delta \in \Gamma_{k+1}(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

$$(A^-)_{\beta\delta} = \begin{cases} A_{\beta\delta} & \text{if } \delta \in \Gamma_{k-1}(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

$$(A^0)_{\beta\delta} = \begin{cases} A_{\beta\delta} & \text{if } \delta \in \Gamma_{k}(\alpha), \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently, for $|k,\beta\rangle$,

$$\begin{aligned} A^+|k,\beta\rangle &= \sum_{\delta\in\Gamma_{k+1}(\alpha),\delta\sim\beta} |k+1,\delta\rangle, \\ A^-|k,\beta\rangle &= \sum_{\delta\in\Gamma_{k-1}(\alpha),\delta\sim\beta} |k-1,\delta\rangle, \\ A^0|k,\beta\rangle &= \sum_{\delta\in\Gamma_{k}(\alpha),\delta\sim\beta} |k,\delta\rangle. \end{aligned} \tag{2.18}$$

Since $\beta \in \Gamma_k(\alpha)$ and $\beta \sim \delta$ then $\delta \in \Gamma_{k-1}(\alpha) \cup \Gamma_k(\alpha) \cup \Gamma_{k+1}(\alpha)$.

It has been shown in [25] that, if $\Lambda(\Gamma)$ is invariant under the quantum components A^+, A^- and A^0, then there exist two sequences (called Szegö–Jacobi sequences) $\{\omega_l\}_{l=1}^{\infty}$ and $\{\alpha_l\}_{l=1}^{\infty}$ derived from A such that

$$\begin{aligned} &A^+|\phi_l\rangle = \sqrt{\omega_{l+1}}|\phi_{l+1}\rangle, \quad l \geq 0, \\ &A^-|\phi_0\rangle = 0, \qquad A^-|\phi_l\rangle = \sqrt{\omega_l}|\phi_{l-1}\rangle, \quad l \geq 1, \\ &A^0|\phi_l\rangle = \alpha_l|\phi_l\rangle, \quad l \geq 0, \end{aligned} \tag{2.19}$$

where $\omega_{l+1} = \frac{\kappa_{l+1}}{\kappa_l}\kappa_-^2(j)$, $\kappa_-(j) = |\{i \in \Gamma_l(\alpha) : i \sim j\}|$ for $j \in \Gamma_{l+1}(\alpha)$ and $\alpha_l = \kappa_0(j)$, such that $\kappa_0(j) = |\{i \in V_l; i \sim j\}|$ for $j \in \Gamma_l(\alpha)$, for $l \geq 0$. One can easily check that the coefficients α_i and ω_i are given by

$$\alpha_k \equiv a_k = \kappa - b_k - c_k, \qquad \omega_k \equiv \beta_k^2 = b_{k-1}c_k, \quad k = 1, \dots, d. \tag{2.20}$$

By using (2.17) and (2.19), one can obtain

$$A|\phi_l\rangle = \beta_{l+1}|\phi_{l+1}\rangle + \alpha_l|\phi_l\rangle + \beta_l|\phi_{l-1}\rangle, \quad l \geq 0, \tag{2.21}$$

with $\beta_l := \sqrt{\omega_l}$. Then, by using (2.21), one can deduce that

$$|\phi_l\rangle = P_l(A)|\phi_0\rangle, \quad l = 1, 2, \dots, d, \tag{2.22}$$

where, $P_l(A) = a_0 + a_1A + a_2A^2 + \cdots + a_lA^l$ is a polynomial of degree l in indeterminate A and conversely A^l can be written as a linear combination of $P_0(A), P_1(A), \dots, P_d(A)$. In fact, it can be shown that [28], the unit vectors $|\phi_l\rangle$ for $l = 0, 1, \dots, d$ are the orthonormal basis produced by application of the orthonormalization process (Lanczos algorithm) to the so called Krylov basis $\{|\phi_0\rangle, A|\phi_0\rangle, A^2|\phi_0\rangle, \dots, A^d|\phi_0\rangle\}$ (for more details see for example [22, 28]).

2.5 Spectral Distribution Associated with the Network

In this subsection we recall some facts about the spectral distribution associated with the adjacency matrix of the network. In fact, the spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics [29, 30]. Since the advent of random matrix theory (RMT), there has been considerable interest in the statistical analysis of spectra [31–33]. Also, the two-point resistance has a probabilistic interpretation based on classical random walker walking on the network. Indeed, the connection between random walks and electric networks has been recognized for some time (see e.g. [34–36]), where one can establish a connection between the electrical concepts of current and voltage and corresponding descriptive quantities of random walks regarded as finite state Markov chains (for more details see [3]). Also, by adapting the random-walk dynamics and mean-field theory it has been studied that [37], how the growth of a conducting network, such as electrical or electronic circuits, interferes with the current flow through the underlying evolving graphs. In [21], it has been shown that, there is also connection between the mathematical techniques introduced in previous subsections and this subsection such as Hilbert space of the stratification and spectral techniques (which have been employed in [22, 28, 38–40] for investigating continuous time quantum walk on graphs), and electrical concept of resistance between two arbitrary nodes of regular networks, and so the same techniques can be used for calculating the resistance. Note that, although we take the spectral approach to obtain an explicit formula for the effective resistance in terms of orthogonal polynomials (which are orthogonal with respect to the spectral distribution μ associated with the network) with three term recursion relations, in practice as it will be seen in Sect. 3, the effective resistances will be calculated without any need to evaluate the spectral distribution μ.

It is well known that, for any pair $(A, |\phi_0\rangle)$ of a matrix A and a vector $|\phi_0\rangle$, it can be assigned a measure μ as follows

$$\mu(x) = \langle\phi_0|E(x)|\phi_0\rangle, \tag{2.23}$$

where $E(x) = \sum_i |u_i\rangle\langle u_i|$ is the operator of projection onto the eigenspace of A corresponding to eigenvalue x, i.e.,

$$A = \int x E(x) dx. \tag{2.24}$$

It is easy to see that, for any polynomial $P(A)$ we have

$$P(A) = \int P(x) E(x) dx, \tag{2.25}$$

where for discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2.23) and (2.25), the expectation value of powers of adjacency matrix A over starting site $|\phi_0\rangle$ can be written as

$$\langle\phi_0|A^m|\phi_0\rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \ldots. \tag{2.26}$$

The existence of a spectral distribution satisfying (2.26) is a consequence of Hamburger's theorem, see e.g., Shohat and Tamarkin [41, Theorem 1.2].

Obviously relation (2.26) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure μ. More clearly, the orthonormality of the unit vectors $|\phi_i\rangle$ implies that

$$\delta_{ij} = \langle\phi_i|\phi_j\rangle = \int_R P_i(x)P_j(x)\mu(dx), \tag{2.27}$$

where, we have used (2.22) to write $|\phi_i\rangle = P_i(A)|\phi_0\rangle$. Now, by substituting (2.22) in (2.21), we get three term recursion relations between polynomials $P_j(A)$, which leads to the following three term recursion relations between polynomials $P_j(x)$

$$xP_k(x) = \beta_{k+1}P_{k+1}(x) + \alpha_k P_k(x) + \beta_k P_{k-1}(x) \tag{2.28}$$

for $k = 0, \dots, d-1$, with $P_0(x) = 1$. Multiplying two sides of (2.28) by $\beta_1 \cdots \beta_k$ we obtain

$$\beta_1 \cdots \beta_k x P_k(x) = \beta_1 \cdots \beta_{k+1} P_{k+1}(x) + \alpha_k \beta_1 \cdots \beta_k P_k(x) + \beta_k^2 . \beta_1 \cdots \beta_{k-1} P_{k-1}(x). \tag{2.29}$$

By rescaling P_k as $Q_k = \beta_1 \cdots \beta_k P_k$, the spectral distribution μ under question is characterized by the property of orthonormal polynomials $\{Q_k\}$ defined recurrently by

$$\begin{aligned} &Q_0(x) = 1, \qquad Q_1(x) = x, \\ &xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \quad k \ge 1 \end{aligned} \tag{2.30}$$

(for more details see [23, 41–43]).

It should be noticed that, the starting values of the recurrence (2.30) are $Q_{-1} = 0$, $Q_0 = 1$. If one starts from $q_{-1} = -1$, $q_0 = 0$ and uses the same recurrence (with $\omega_0 = 1$)

$$xq_n(x) = q_{n+1}(x) + \alpha_n q_n(x) + \omega_n q_{n-1}(x), \tag{2.31}$$

then q_n is of degree $n-1$, and by Favard's theorem the different q_n's are orthogonal with respect to some measure. The q_n's are called orthogonal polynomials of the second kind (sometimes for Q_n we say that they are of the first kind). They can also be written in the form

$$q_n(z) = \int_R \frac{Q_m(z) - Q_m(x)}{z - x} d\mu(x), \tag{2.32}$$

(for more details see for example [44]).

3 Evaluation of Effective Resistances in Pseudo-Distance-Regular Resistor Networks

In this section, we consider pseudo-distance-regular graphs as resistor networks and obtain an explicit formula for evaluation of effective resistances. The results obtained in this section show that, there is a close connection between the techniques introduced in Sect. 2 such as Hilbert space of the stratification and the orthogonal polynomials of the first and second kind associated with the networks, and electrical concept of resistance between two arbitrary nodes of the networks.

Hereafter, we assume that all nonzero resistances (associated with the edges) are equal to 1. Then, by using (2.3), the off-diagonal elements of $-L$ are precisely those of the corresponding adjacency matrix A, i.e., we have

$$L = \kappa I - A, \tag{3.33}$$

with $\kappa \equiv \kappa_1 = \deg(\alpha)$, for each vertex α.

Now, consider two nodes $\alpha, \beta \in V$ such that $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ (as we will see in Sect. 4.1, for distance-regular graphs as resistor networks, the diagonal entries of L^{-1} are independent of the vertex, i.e., for all $\alpha, \beta \in V$, we have $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$). Then, from the relation (2.6) and the fact that L^{-1} is a real matrix, we can obtain the two-point resistance between the nodes α and β as follows

$$R_{\alpha\beta} = 2(L^{-1}_{\alpha\alpha} - L^{-1}_{\alpha\beta}). \tag{3.34}$$

It should be noticed that, due to the stratification of the network, all of the nodes belonging to the same stratum with respect to the reference node (a node which stratification is done with respect to it), possess the same effective resistance with respect to the reference node (the proof is given in the Appendix). More clearly, in order to evaluate the effective resistance between two nodes α and β of a network, we consider one of the nodes, say α, as reference node and stratify the network with respect to α. Then, β will belong to one of the strata with respect to α, say the m-th stratum $\Gamma_m(\alpha)$. Now, for all $\beta \in \Gamma_m(\alpha)$, one can write

$$L^{-1}_{\alpha\beta^{(m)}} = \langle\alpha|L^{-1}|\beta\rangle = \frac{1}{\sqrt{\kappa_m}}\langle\alpha|L^{-1}|\phi_m\rangle = \frac{1}{\sqrt{\kappa_m}}\langle\alpha|P_m(A)L^{-1}|\alpha\rangle, \tag{3.35}$$

where, we have used (2.11) and (2.22) and the lemma given in the Appendix. Then, by using (3.34), we obtain for all $\beta \in \Gamma_m(\alpha)$

$$\begin{aligned} R_{\alpha\beta^{(m)}} &= \frac{2}{\sqrt{\kappa_m}}\left\{\sqrt{\kappa_m}L^{-1}_{\alpha\alpha} - (P_m(A)L^{-1})_{\alpha\alpha}\right\} \\ &= \frac{2}{\sqrt{\kappa_m\omega_1\cdots\omega_m}}\langle\alpha|\frac{\sqrt{\kappa_m\omega_1\cdots\omega_m}1 - Q_m(A)}{\kappa 1 - A}|\alpha\rangle \\ &= \frac{2}{\sqrt{\kappa_m\omega_1\cdots\omega_m}}\int_{R-\{\kappa\}}\frac{Q_m(\kappa) - Q_m(x)}{\kappa - x}d\mu(x), \end{aligned} \tag{3.36}$$

where, the upper index m in $L^{-1}_{\alpha\beta^{(m)}}$ and $R_{\alpha\beta^{(m)}}$ indicate that β belongs to the m-th stratum with respect to α. Note that, we have substituted $P_m(x) = \frac{1}{\sqrt{\omega_1\cdots\omega_m}}Q_m(x)$ and used the equality $Q_m(\kappa) = \sqrt{\kappa_m\omega_1\cdots\omega_m}$ which can be verified easily. It should be noticed that, the result (3.36) can be written as

$$R_{\alpha\beta^{(m)}} = \frac{2}{\sqrt{\kappa_m\omega_1\cdots\omega_m}}\left\{\int_R\frac{Q_m(\kappa) - Q_m(x)}{\kappa - x}d\mu(x) - \frac{1}{\upsilon}\left(\frac{\partial}{\partial x}Q_m(x)\right)\bigg|_{x=\kappa}\right\}. \tag{3.37}$$

Now, by using (2.32) and (3.37), we obtain the main result of the paper as follows

$$R_{\alpha\beta^{(m)}} = \frac{2}{\sqrt{\kappa_m\omega_1\cdots\omega_m}}\left\{q_m(\kappa) - \frac{1}{\upsilon}\left(\frac{\partial}{\partial x}Q_m(x)\right)\bigg|_{x=\kappa}\right\}, \quad m = 1, 2, \ldots, d. \tag{3.38}$$

We recall that, the result (3.38) can be used only for evaluating the effective resistance between nodes such that the network is symmetric with respect to them in the sense that, stratification of the network with respect to these nodes produces the same strata. Then for such nodes α and β, we will have $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$.

4 Examples

4.1 Evaluation of Effective Resistances on Distance-Regular Networks

First, we recall the definition of special kind of pseudo-distance-regular networks called distance-regular networks:

Definition A pseudo-distance regular network $\Gamma = (V, E)$ is called distance-regular with diameter d if for all $k \in \{0, 1, \dots, d\}$, and $\alpha, \beta \in V$ with $\beta \in \Gamma_k(\alpha)$, the numbers $c_k(\beta)$, $a_k(\beta)$ and $b_k(\beta)$ defined in (2.13) depend only on k but do not depend on the choice of α and β.

It should be noticed that, in distance-regular networks, the i-th adjacency matrix of the network $\Gamma = (V, R)$ is defined by

$$\left(A_i\right)_{\alpha,\beta} = \begin{cases} 1 & \text{if } \partial(\alpha, \beta) = i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.39}$$

Then, from the definition of A_i, for the reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), we have

$$A_i|\phi_0\rangle = \sum_{\beta \in \Gamma_i(o)} |\beta\rangle. \tag{4.40}$$

Then by using (2.11) and (4.40), we have

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{4.41}$$

Also, it can be shown that, for adjacency matrices of a distance regular graph, we have

$$\begin{aligned} &A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}, \quad \text{for } i = 1, 2, \dots, d-1, \\ &A_1 A_d = b_{d-1} A_{d-1} + a_d A_d. \end{aligned} \tag{4.42}$$

Using the recursion relations (4.42), one can show that A_i is a polynomial in A_1 of degree i, i.e., we have

$$A_i = P_i(A_1), \quad i = 1, 2, \dots, d, \tag{4.43}$$

and conversely A_1^i can be written as a linear combination of $I, A_1, \dots, A_d$ (for more details see for example [22]).

Now, it should be noticed that, stratification of distance-regular networks will be independent of the choice of the reference node (the node which stratification is done with respect to it). Then, clearly we will have $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ for all $\alpha, \beta \in V$ with $\alpha \neq \beta$ and consequently, the result (3.38) can be used for evaluation of the effective resistance between any two arbitrary nodes α, β; It is sufficient to choose one of these nodes, say α, as reference node and stratify the network with respect to it. Then, β will be contained in one of the strata, say m-th stratum, with respect to α and so the effective resistance between $\alpha, \beta \in \Gamma_m(\alpha)$ can be evaluated via (3.38).

4.1.1 Cycle Network C_v

A well known example of distance-regular networks, is the cycle network C_v with $\kappa = 2$. The network C_v for $v = 2k$ or $v = 2k + 1$ consists of $k + 1$ strata, where the intersection arrays for even and odd number of vertices are given by

$$\{b_0, \dots, b_{k-1}; c_1, \dots, c_k\} = \{2, 1, \dots, 1, 1; 1, \dots, 1, 2\} \quad \text{and} \tag{4.44}$$

$$\{b_0, \dots, b_{k-1}; c_1, \dots, c_k\} = \{2, 1, \dots, 1; 1, \dots, 1, 1\}, \tag{4.45}$$

respectively. Then, by using (2.20), for even $v = 2k$ the QD parameters are given by

$$\alpha_i = 0, \quad i = 0, 1, \dots, k; \qquad \omega_1 = \omega_k = 2, \qquad \omega_i = 1, \quad i = 2, \dots, k-1, \tag{4.46}$$

where, for odd $v = 2k + 1$, we obtain

$$\alpha_i = 0, \quad i = 0, 1, \dots, k-1, \qquad \alpha_k = 1; \qquad \omega_1 = 2, \qquad \omega_i = 1, \quad i = 2, \dots, k. \tag{4.47}$$

We consider $v = 2k$ (the case $v = 2k + 1$ can be considered similarly). Then, by using (2.30) and (4.46), one can obtain

$$Q_0 = 1, \qquad Q_1 = x, \qquad Q_i(x) = 2T_i(x/2), \quad i > 1; \qquad q_i(x) = Q_{i-1}(x), \quad i \geq 1, \tag{4.48}$$

where, T_i's are Chebyshev polynomials in one variable, which are recursively defined by

$$T_0 = 1, \qquad T_1 = x, \qquad T_n(x) = 2xT_{n-1} - T_{n-2}, \quad n > 1. \tag{4.49}$$

Then, by using (3.38) and (4.48), the effective resistance between any two nodes $\alpha, \beta \in \Gamma_m(\alpha)$, is obtained as

$$R_{\alpha\beta^{(1)}} = q_1(2) - \frac{1}{2k}\frac{\partial}{\partial x}Q_1(x)\bigg|_{x=2} = 1 - \frac{1}{2k},$$

$$R_{\alpha\beta^{(m)}} = 2T_{m-1}(1) - \frac{1}{2k}T'_m(1), \quad \text{for } m > 1. \tag{4.50}$$

4.1.2 $(d + 1)$-Simplex Fractals

$(d + 1)$-simplex fractal is a generalization of a two dimensional Sierpinski gasket to d-dimensions such that its subfractals are $(d + 1)$-simplices or d-dimensional polyhedra with $S_{(d+1)}$-symmetry. In order to obtain a fractal with decimation number 2, we choose a $(d + 1)$-simplex and divide all the links (that is the lines connecting sites) into 2 parts and then draw all possible d-dimensional hyperplanes through the links parallel to the transverse d-simplices. Next, having omitted every other innerpolyhedra, we repeat this process for the remaining simplices or for the subfractals of next higher generation. This way through $(d + 1)$-simplex fractals are constructed. We label subfractals of generation $(d + 1)$ in terms of partition of 1 into $(d + 1)$ positive integers $\lambda_1, \lambda_2, \dots, \lambda_{d+1}$. Each partition represents a subfractal of generation d, and λ shows the distance of the corresponding subfractal from d-dimensional hyperplanes which construct the $(d + 1)$-simplex. On the other hand, each vertex denoted by partition of 2 into $(d + 1)$ non-negative integers $\eta_1, \eta_2, \dots, \eta_{d+1}$ and obviously the i-th vertex of subfractal $(\lambda_1, \lambda_2, \dots, \lambda_{d+1})$ is denoted by $\eta_j = \lambda_j + \delta_{i,j}$, where $j = 1, 2, \dots, d + 1$ (for more details see [45, 46]).

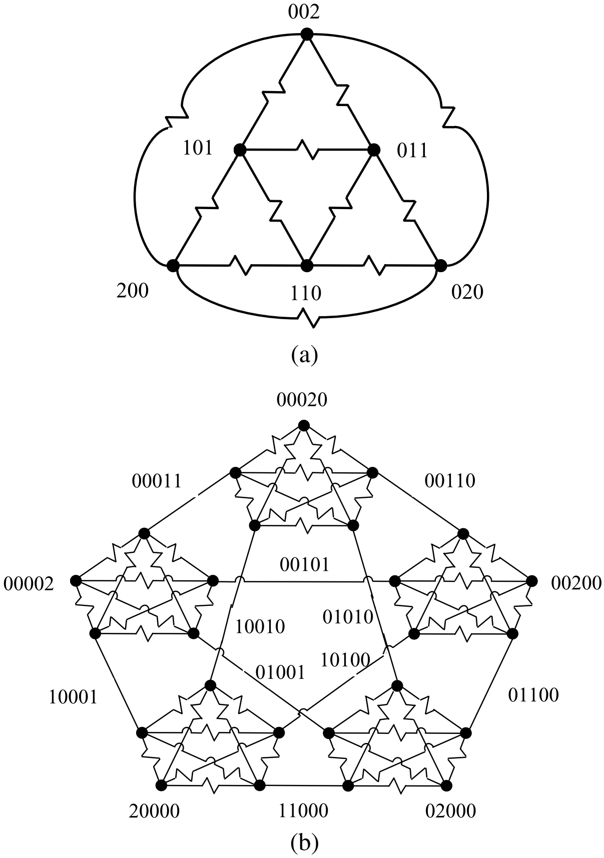

Fig. 2 (**a**) Shows 3-simplex fractal with decimation number $b=2$. (**b**) Shows 5-simplex fractal with decimation number $b=2$

Now, we consider the $(d+1)$-simplex fractal with decimation number $b=2$ such that all of the $d+1$ vertices $\underbrace{(20\ldots0)}_{d+1}, \underbrace{(020\ldots0)}_{d+1}, \ldots, \underbrace{(0\ldots02)}_{d+1}$ are connected via a resistor to each other (see Fig. 2(a) and (b) for $d=2$ and $d=4$, respectively). Then, the number of vertices o $(d+1)$-simplex fractal is $v = C_1^{d+1} + C_2^{d+1} = \frac{(d+1)(d+2)}{2}$ such that the degree of each vertex is $\kappa = 2d$. Also, it can be easily shown that the network has 3 strata with respect to the reference node $(200\ldots0)$, where the unit vectors $|\phi_i\rangle$, $i=0,1,2$ are given as follows

$$|\phi_0\rangle = \underbrace{|200\ldots0\rangle}_{d+1},$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2d}} \sum_{i=2}^{d+1} (|00\ldots0\underbrace{2}_{i}0\ldots0\rangle + |10\ldots0\underbrace{1}_{i}0\ldots0\rangle), \tag{4.51}$$

$$|\phi_2\rangle = \sqrt{\frac{2}{d(d-1)}} \sum_{i,j=2,\ldots,d+1; i<j} |0\ldots0\underbrace{1}_{i}0\ldots0\underbrace{1}_{j}0\ldots0\rangle.$$

Then, one can show that

$$\begin{aligned} A|\phi_0\rangle &= \sqrt{2d}|\phi_1\rangle, \\ A|\phi_1\rangle &= \sqrt{2d}|\phi_0\rangle + d|\phi_1\rangle + 2\sqrt{d-1}|\phi_2\rangle, \\ A|\phi_2\rangle &= 2\sqrt{d-1}|\phi_1\rangle + 2(d-2)|\phi_2\rangle. \end{aligned} \tag{4.52}$$

Also, by using the recursion relations (4.52) one can easily show that the adjacency matrices $A \equiv A_1$, $A_2 = J - A - I$ (where, J is all one matrix) satisfy the following relations

$$\begin{aligned} A^2 &= 2d.I_{\frac{(d+1)(d+2)}{2}} + d.A + 4A_2, \\ AA_2 &= (d-1)A + 2(d-2)A_2. \end{aligned} \tag{4.53}$$

It should be noticed that if we stratify the network with respect to another reference node such as $(1100\ldots0)$, the unit vectors will be obtained as

$$\begin{aligned} |\phi_0\rangle &= \underbrace{|1100\ldots0\rangle}_{d+1}, \\ |\phi_1\rangle &= \frac{1}{\sqrt{2d}}\Big\{\sum_{i=3,\ldots,d+1}(|10\ldots0\underbrace{1}_{i}0\ldots0\rangle + |010\ldots0\underbrace{1}_{i}0\ldots0\rangle) \\ &\quad + |200\ldots0\rangle + |020\ldots0\rangle\Big\}, \\ |\phi_2\rangle &= \sqrt{\frac{2}{d(d-1)}}\Big\{\sum_{i=3,\ldots,d+1}|00\ldots0\underbrace{2}_{i}0\ldots0\rangle \\ &\quad + \sum_{i,j=3,\ldots,d+1;i<j}|00\ldots0\underbrace{1}_{i}0\ldots0\underbrace{1}_{j}0\ldots0\rangle\Big\}. \end{aligned} \tag{4.54}$$

Then, one can show that, the same recursion relations as in (4.52) and (4.53) are satisfied for this stratification, i.e., stratification of the network gives three-term recursion relations independent of the choice of the reference node and so the network is distance-regular.

Now, by using (2.21) and (4.52), one can obtain

$$\alpha_0 = 0, \qquad \alpha_1 = d, \qquad \alpha_2 = 2(d-2); \qquad \omega_1 = 2d, \qquad \omega_2 = 4(d-1). \tag{4.55}$$

Then, by using the recursion relations (2.30) and (2.31), one can see that

$$Q_1(x) = x, \qquad Q_2(x) = x^2 - dx - 2d; \qquad q_1(x) = 1, \qquad q_2(x) = x - d. \tag{4.56}$$

It should be noticed that, the intersection array of the $(d+1)$-simplex fractal with decimation number $b = 2$, can be evaluated by using (2.20) and (4.55) as: $\{b_0, b_1; c_1, c_2\} = \{2d, d-1; 1, 4\}$ which implies that $\kappa = b_0 = 2d$, $\kappa_2 = \frac{\kappa b_1}{c_2} = \frac{d(d-1)}{2}$. Now, by using (3.38) and (4.56), the effective resistance between any node $\alpha \in V$ and $\beta \in \Gamma_m(\alpha)$ for $m = 1, 2$ is given by

$$R_{\alpha,\beta^{(1)}} = \frac{2}{\kappa}\left(1 - \frac{1}{v}\right) = \frac{1}{d}\left(1 - \frac{2}{(d+1)(d+2)}\right) = \frac{d+3}{(d+1)(d+2)},$$

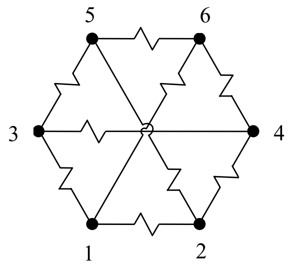

Fig. 3 Shows the Hexagon network

$$R_{\alpha,\beta^{(2)}} = \frac{1}{d(d-1)}\left\{q_2(2d) - \frac{1}{v}(2x-d)|_{x=2d}\right\}$$
$$= \frac{1}{d(d-1)}\left\{d - \frac{2}{(d+1)(d+2)}.3d\right\} = \frac{d+4}{(d+1)(d+2)}. \tag{4.57}$$

4.1.3 Hexagon Network

Consider the hexagon network with its diameters shown in Fig. 3. As the figure shows, this network has 6 nodes with intersection array $\{b_0, b_1; c_1, c_2\} = \{3, 2; 1, 3\}$. Then by using (2.16) and (2.20), one can obtain

$$\kappa = 3, \qquad \kappa_2 = 2; \qquad \alpha_0 = \alpha_1 = \alpha_2 = 0, \qquad \omega_1 = 3, \qquad \omega_2 = 6. \tag{4.58}$$

Then, by using (2.30) and (4.58), we obtain

$$Q_0 = 1, \qquad Q_1 = x, \qquad Q_2(x) = x^2 - 3; \qquad q_1(x) = 1, \qquad q_2(x) = x. \tag{4.59}$$

Now, by using (3.38) and (4.59), the effective resistance between any node $\alpha \in V$ and $\beta \in \Gamma_m(\alpha)$ for $m = 1, 2$ is given by

$$R_{\alpha,\beta^{(1)}} = \frac{2}{3}\left(1 - \frac{1}{6}\right) = \frac{5}{9}, \qquad R_{\alpha,\beta^{(2)}} = \frac{1}{3}\left\{q_2(3) - \frac{1}{6}(2x)|_{x=3}\right\} = \frac{2}{3}. \tag{4.60}$$

4.1.4 A Bipartite Distance-Regular Network with $2n$ Nodes

Consider a distance-regular network with $2n$ nodes and adjacency matrices as

$$A_1 = \sigma_x \otimes (J_n - I_n), \qquad A_2 = I_2 \otimes (J_n - I_n), \qquad A_3 = \sigma_x \otimes I_n, \tag{4.61}$$

where, J_n is an $n \times n$ matrix such that all of its entries are one. Then, one can show that

$$A_1^2 = (n-1)I_{2n} + (n-2)A_2, \qquad A_1A_2 = (n-2)A_1 + (n-1)A_3, \qquad A_1A_3 = A_2. \tag{4.62}$$

By using (4.42) and (4.62), the intersection array of the network is given by

$$\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{n-1, n-2, 1; 1, n-2, n-1\}.$$

Then, by using (2.16) and (2.20), one can obtain

$$\kappa = n-1, \qquad \kappa_2 = n-1, \qquad \kappa_3 = 1,$$
$$\alpha_i = 0, \quad i = 0, 1, 2, \qquad \omega_1 = n-1, \qquad \omega_2 = (n-2)^2, \qquad \omega_3 = n-1. \tag{4.63}$$

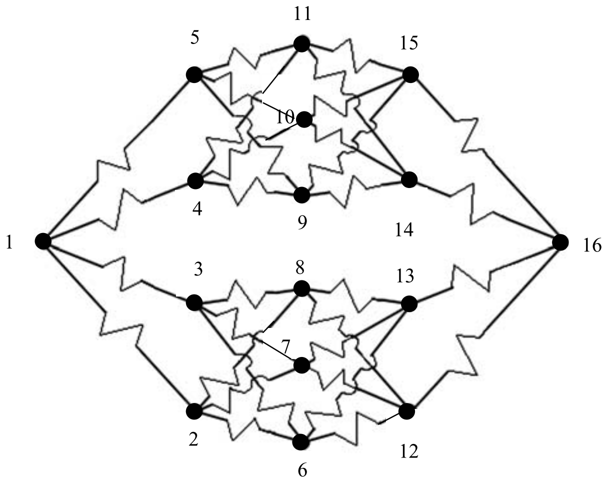

Fig. 4 Shows a pseudo-distance-regular network with 16 nodes derived from Hadamard graph

Now, by using (2.30) and (4.63), we obtain

$$Q_0 = 1, \qquad Q_1 = x, \qquad Q_2(x) = x^2 - n + 1, \qquad Q_3(x) = x^3 - (n^2 - 3n + 3)x;$$
$$q_1(x) = 1, \qquad q_2(x) = x, \qquad q_3(x) = x^2 - (n-2)^2. \tag{4.64}$$

So, by using (3.38) and (4.64), the effective resistance between any node $\alpha \in V$ and $\beta \in \Gamma_m(\alpha)$ for $m = 1, 2, 3$ is given by

$$\begin{aligned} R_{\alpha,\beta^{(1)}} &= \frac{2}{n-1}\left(1 - \frac{1}{2n}\right) = \frac{2n-1}{n(n-1)}, \\ R_{\alpha,\beta^{(2)}} &= \frac{2}{(n-1)(n-2)}\left\{q_2(n-1) - \frac{1}{2n}(2x)|_{x=n-1}\right\} = \frac{2(n-1)}{n(n-2)}, \\ R_{\alpha,\beta^{(3)}} &= \frac{2}{(n-1)(n-2)}\left\{q_3(n-1) - \frac{1}{2n}[3x^2 - (n^2 - 3n + 3)]|_{x=n-1}\right\} = \frac{2n-3}{(n-1)(n-2)}. \end{aligned} \tag{4.65}$$

4.2 Evaluation of Effective Resistance in Examples of Pseudo-Distance-Regular Networks

4.2.1 Pseudo-Distance-Regular Network Derived from Hadamard Network with 16 *Nodes*

Consider the pseudo-distance-regular network shown in Fig. 4. This network is obtained from the Hadamard network with intersection array $\{4, 3, 2, 1; 1, 2, 3, 4\}$. As Fig. 4 shows, the network is symmetric with respect to the initial and final (horizontal) nodes 1 and $16 \in \Gamma_4(1)$ and also with respect to the initial and final (vertical) nodes 6 and $11 \in \Gamma_4(6)$. One should notice that stratification of the network with respect to the nodes 1 and 16 produces the same strata. For stratification with respect to the node 1 or 16, we have

$$\kappa = 4, \qquad \kappa_2 = 6, \qquad \kappa_3 = 4, \qquad \kappa_4 = 1,$$
$$\alpha_i = 0, \quad i = 0, \dots, 3; \qquad \omega_1 = 4, \qquad \omega_2 = 6, \qquad \omega_3 = 6, \qquad \omega_4 = 4$$

(see (2.19)). Then, by using the recursion relations (2.30) and (2.31), one can obtain

$$Q_4(x) = x^4 - 16x^2 + 24, \qquad q_4(x) = x^3 - 12x. \tag{4.66}$$

Now, from (3.38), the effective resistance between nodes 1 and $16 \in \Gamma_4(1)$ is given by

$$R_{1,16} = \frac{1}{12}\left\{q_4(4) - \frac{1}{16}(4x^3 - 32x)|_{x=4}\right\} = \frac{2}{3}. \tag{4.67}$$

In order to evaluate the effective resistance between vertical nodes 6 and 11, one must consider the stratification of the network with respect to the node 6 or 11. For this stratification, we obtain

$$\kappa = 4, \qquad \kappa_2 = 4, \qquad \kappa_3 = 4, \qquad \kappa_4 = 3$$
$$\alpha_i = 0, \quad i = 0, \dots, 3; \qquad \omega_1 = 4, \qquad \omega_2 = 4, \qquad \omega_3 = 1, \qquad \omega_4 = 12.$$

Then, by using the recursion relations (2.30) and (2.31), one can obtain

$$Q_4(x) = x^4 - 9x^2 + 4, \qquad q_4(x) = x^3 - 5x. \tag{4.68}$$

Therefore, from (3.38), the effective resistance between nodes 6 and $11 \in \Gamma_4(6)$ is given by

$$R_{6,11} = \frac{1}{12}\left\{q_4(4) - \frac{1}{16}(4x^3 - 18x)|_{x=4}\right\} = \frac{65}{24}. \tag{4.69}$$

4.2.2 Pseudo-Distance-Regular Network Derived from Desargues

This network has 20 nodes with

$$\{b_0, b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5\} = \{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}.$$

Then, by using (2.16) and (2.20), one can obtain

$$\kappa = 3, \qquad \kappa_2 = 6, \qquad \kappa_3 = 6, \qquad \kappa_4 = 3, \qquad \kappa_5 = 1,$$
$$\alpha_i = 0, \quad i = 0, 1, \dots, 5; \qquad \omega_1 = 3, \qquad \omega_2 = 2, \qquad \omega_3 = 4, \qquad \omega_4 = 2, \qquad \omega_5 = 3.$$

The stratification with respect to the initial node 1 and the final node 20 produces the same strata. Therefore, the effective resistance between the nodes 1 and 20 can be evaluated by using (3.38). By using the recursion relations (2.30) and (2.31), we obtain

$$Q_5(x) = x^5 - 11x^3 + 22x, \qquad q_5(x) = x^4 - 8x^2 + 4. \tag{4.70}$$

Then, from (3.38), the effective resistance between nodes 1 and $20 \in \Gamma_5(1)$ is given by

$$R_{1,20} = \frac{1}{6}\left\{q_5(3) - \frac{1}{20}(5x^4 - 33x^2 + 22)|_{x=3}\right\} = \frac{13}{12}. \tag{4.71}$$

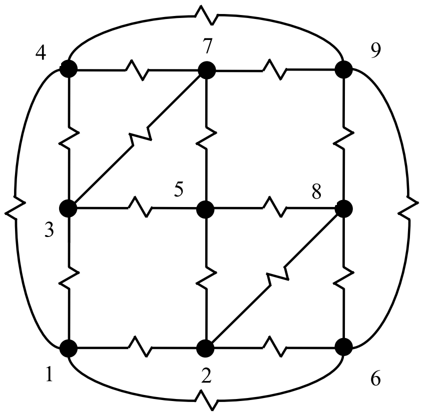

Fig. 5 Shows a pseudo-distance-regular network with 9 nodes

4.2.3 Pseudo-Distance-Regular Network Derived from Hadamard Network with 32 *Nodes*

Consider the pseudo-distance-regular network obtained from the Hadamard network with intersection array $\{8, 7, 4, 1; 1, 4, 7, 8\}$ such that, the network is symmetric with respect to the initial and final (horizontal) nodes 1 and $32 \in \Gamma_4(1)$ and also with respect to the initial and final (vertical) nodes 10 and $23 \in \Gamma_4(10)$. One should notice that stratification of the network with respect to the nodes 1 and 32 produces the same strata. For stratification with respect to the node 1 or 32, we have

$$\kappa = 8, \qquad \kappa_2 = 14, \qquad \kappa_3 = 8, \qquad \kappa_4 = 1,$$
$$\alpha_i = 0, \quad i = 0, \ldots, 4; \qquad \omega_1 = 8, \qquad \omega_2 = 28, \qquad \omega_3 = 28, \qquad \omega_4 = 8$$

(see (2.19)). Then, by using the recursion relations (2.30) and (2.31), one can obtain

$$Q_4(x) = x^4 - 64x^2 + 224, \qquad q_4(x) = x^3 - 56x. \tag{4.72}$$

Now, from (3.38), the effective resistance between nodes 1 and $32 \in \Gamma_4(1)$ is given by

$$R_{1,32} = \frac{1}{112}\left\{q_4(8) - \frac{1}{32}(4x^3 - 128x)|_{x=8}\right\} = \frac{2}{7}. \tag{4.73}$$

In order to evaluate the effective resistance between vertical nodes 10 and $23 \in \Gamma_4(10)$, one must consider the stratification of the network with respect to the node 10 or 23 and evaluate $R_{10,23}$ as before.

4.2.4 A Network with 9 *Nodes*

Consider the network given in Fig. 5 with 9 nodes and the following intersection array

$$\{b_0, b_1; c_1, c_2\} = \{4, 2; 1, 2\}.$$

Then by using (2.16) and (2.20), one can obtain

$$\kappa = 4, \qquad \kappa_2 = 4; \qquad \alpha_0 = 0, \qquad \alpha_1 = 1, \qquad \alpha_2 = 2, \qquad \omega_1 = \omega_2 = 4.$$

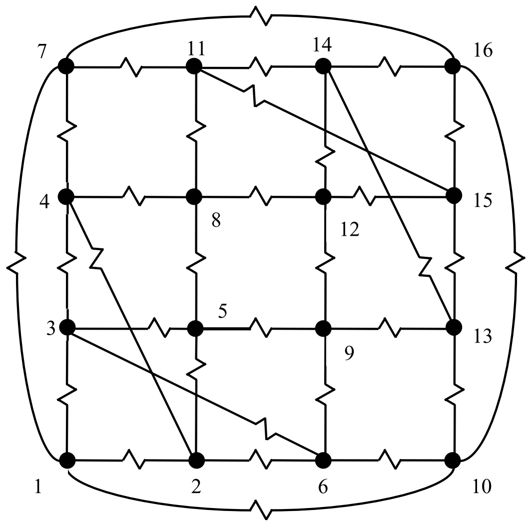

Fig. 6 Shows a pseudo-distance-regular network with 16 nodes

As it can be seen from Fig. 5, the stratification with respect to the nodes 1 and $9 \in \Gamma_2(1)$ produces the same strata. Then, by using the recursion relations (2.30) and (2.31), one can obtain

$$Q_2(x) = x^2 - x - 4, \qquad q_2(x) = x - 1. \tag{4.74}$$

Now, from (3.38), the effective resistance between nodes 1 and $9 \in \Gamma_2(1)$ is given by

$$R_{1,9} = \frac{1}{4}\left\{q_2(4) - \frac{1}{9}(2x-1)|_{x=4}\right\} = \frac{5}{9}. \tag{4.75}$$

4.2.5 A Network with 16 *Nodes*

Consider the network given in Fig. 6 with 16 nodes and the following intersection array

$$\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{4, 3, 2, 1; 1, 2, 3, 4\}.$$

Then by using (2.16) and (2.20), one can obtain

$$\kappa = 4, \qquad \kappa_2 = 6, \qquad \kappa_3 = 4, \qquad \kappa_4 = 1;$$
$$\alpha_i = 0, \quad i = 0, 1, \ldots, 4, \qquad \omega_1 = 4, \qquad \omega_2 = \omega_3 = 6, \qquad \omega_4 = 4.$$

As it can be seen from Fig. 6, the stratification with respect to the nodes 1 and $16 \in \Gamma_2(1)$ produces the same strata. Then, by using the recursion relations (2.30) and (2.31), one can obtain

$$Q_2(x) = x^2 - 4, \qquad q_2(x) = x. \tag{4.76}$$

Now, from (3.38), the effective resistance between nodes 1 and $16 \in \Gamma_2(1)$ is given by

$$R_{1,16} = \frac{1}{6}\left\{q_2(4) - \frac{1}{16}(2x)|_{x=4}\right\} = \frac{7}{12}. \tag{4.77}$$

4.2.6 Generalized $G_2(m)$ Type Network

Consider the network with $2(2^{m+1}-1)$ nodes and the following intersection array

$$\{b_0, b_1, \dots, b_m; c_1, c_2, \dots, c_m, c_{m+1}\} = \{3, 2, \dots, 2; 1, 1, \dots 1, 3\}.$$

Then by using (2.16) and (2.20), one can obtain

$$\kappa = 3, \qquad \kappa_i = 3.2^{i-1}, \quad i = 2, \dots, m, \qquad \kappa_{m+1} = 2^m,$$
$$\alpha_i = 0, \quad i = 0, 1, \dots, m+1; \qquad \omega_1 = 3, \quad \omega_2 = \dots = \omega_m = 2, \qquad \omega_{m+1} = 6.$$

Then, the stratification with respect to the initial node 1 and the final node $2^{m+2}-2 \in \Gamma_1(1)$ produces the same strata. Therefore, from the fact that $Q_1(x) = x$ and $q_1(x) = 1$, the effective resistance between nodes 1 and $2^{m+2}-2 \in \Gamma_1(1)$ is obtained as

$$R_{1,2^{m+2}-2} = \frac{2}{3}\left(1 - \frac{1}{2(2^{m+1}-1)}\right) = \frac{2^{m+2}-3}{3(2^{m+1}-1)}. \tag{4.78}$$

5 Conclusion

Based on the stratification of the pseudo-distance-regular networks and using spectral techniques, the evaluation of the effective resistances on these networks was discussed. It was shown that, in these types of networks, the effective resistances between a node α and all nodes β belonging to the same stratum with respect to the α are the same. Then, an explicit analytical formula for the effective resistance between two nodes α, β of a pseudo-distance-regular resistor network such that $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ (L^{-1} is the pseudo-inverse of the Laplacian matrix of the network) was given in terms of the first and second orthogonal polynomials associated with the network. It was deduced that, the obtained result can be used for evaluation of the effective resistance between any two arbitrary nodes α, β in distance-regular networks, where we have $L^{-1}_{\alpha\alpha} = L^{-1}_{\beta\beta}$ for all nodes α, β.

Appendix

In this appendix we prove the following lemma in connection with the equality of effective resistance between the reference node and all of the nodes belonging to the same stratum with respect to the reference node.

Lemma *Let $R_{\alpha\beta}$ denote the effective resistance between nodes $\alpha, \beta \in V$. Then for pseudo-distance-regular networks, by choosing one of the nodes, say α as reference node, the effective resistance $R_{\alpha\beta}$ is the same for all nodes $\beta \in \Gamma_m(\alpha)$, where $m \in \{1, 2, \dots, d\}$.*

Proof In order to prove the above lemma, we prove that in general for any pseudo-distance-regular network with diameter d, we have $\langle\phi_0|f(A)|\beta\rangle$ ($|\phi_0\rangle \equiv |\alpha\rangle$) is the same for all $\beta \in \Gamma_l(\alpha)$, i.e.,

$$\langle\phi_0|f(A)|\beta\rangle = \frac{1}{\sqrt{\kappa_l}}\langle\phi_0|f(A)|\phi_l\rangle, \tag{A.1}$$

where, $f(A)$ is any function of the adjacency matrix A of the network such that $f(A) = \sum_{l=0}^{d} a_k A^k$ and

$$|\phi_l\rangle = \frac{1}{\sqrt{\kappa_l}} \sum_{j \in \Gamma_l(\alpha)} |j\rangle, \quad l = 0, 1, \dots, d. \tag{A.2}$$

To this aim, we take the Fourier transform of the unit vectors $|\phi_l\rangle$ for $l = 0, 1, \dots, d$ as follows

$$|\phi_{l,k}\rangle = \frac{1}{\sqrt{\kappa_l}} \sum_{j \in \Gamma_l(\alpha)} e^{2\pi i j k/\kappa_l} |j\rangle, \quad l = 0, 1, \dots, d;\ k = 0, 1, \dots, \kappa_l - 1. \tag{A.3}$$

Now, we show that

$$\begin{aligned} &\langle \phi_0 | f(A) | \phi_{l,0} \rangle \neq 0, \\ &\langle \phi_0 | f(A) | \phi_{l,k} \rangle = 0, \quad \forall k = 1, \dots, \kappa_l - 1. \end{aligned} \tag{A.4}$$

To do so, we use the fact that there is a correspondence between the basis $I, A, \dots, A^{d-1}$ and the orthogonal polynomials $P_i(A)$ defined by (4.43). In fact, as it was regarded previously (see arguments about (4.43)), A^l for $l = 0, 1, \dots, d$ can be written as a linear combination of $P_0(A), P_1(A), \dots, P_d(A)$.

It should be noticed that, in Krylov subspace projection methods, approximations to the desired eigenpairs of an $n \times n$ matrix A are extracted from a d-dimensional Krylov subspace

$$K_d(|\phi_0\rangle, A) = \text{span}\{|\phi_0\rangle, A|\phi_0\rangle, \dots, A^{d-1}|\phi_0\rangle\}, \tag{A.5}$$

where, $|\phi_0\rangle$ is often a randomly chosen starting vector called reference state and $d \ll n$, i.e., the vectors $|\phi_0\rangle, A|\phi_0\rangle, \dots, A^{d-1}|\phi_0\rangle$ constitute a basis for the Krylov subspace $K_d(|\phi_0\rangle, A)$. Then, the application of the orthonormalization process (the Lanczos algorithm which is a modified version of the classical Gram-Schmidt orthogonalization process) to the Krylov basis $\{A^k|\phi_0\rangle\}_{k=0}^{d-1}$ is equivalent to the construction of a sequence of orthonormal basis $|\phi_j\rangle = P_j(A)|\phi_0\rangle$, where $P_j(A) = a_0 + a_1 A + \dots + a_j A^j$ is a polynomial of degree j in indeterminate A.

As regards the above arguments, any function $f(A)$ can be expanded as a linear combination of the polynomials $P_j(A)$, i.e.,

$$f(A) = \sum_{j=0}^{d} b_j P_j(A). \tag{A.6}$$

Then, we have

$$\langle \phi_0 | f(A) | \phi_{l,k} \rangle = \sum_{j=0}^{d} b_j \langle \phi_0 | P_j(A) | \phi_{l,k} \rangle = \sum_{j=0}^{d} b_j \underbrace{\langle \phi_{j,0} | \phi_{l,k} \rangle}_{\delta_{jl}\delta_{k0}} = 0, \quad \forall k = 1, \dots, \kappa_l - 1. \tag{A.7}$$

Now, let we denote $\langle \phi_0 | f(A) | j \rangle$ by x_j for $j \in \Gamma_l(\alpha)$. Then, from (A.7) we have

$$F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\kappa_l} \end{pmatrix} = \begin{pmatrix} \langle \phi_0 | f(A) | \phi_{l,0} \rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{A.8}$$

where, F is the $\kappa_l \times \kappa_l$ discrete Fourier transformation matrix(DFT) defined as $F_{jk} = \frac{1}{\sqrt{\kappa_l}} e^{2\pi i jk/\kappa_l}$. Therefore, by inverting F in (A.8), we obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\kappa_l} \end{pmatrix} = F^{\dagger} \begin{pmatrix} \langle \phi_0 | f(A) | \phi_{l,0} \rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\kappa_l}} \begin{pmatrix} \langle \phi_0 | f(A) | \phi_{l,0} \rangle \\ \langle \phi_0 | f(A) | \phi_{l,0} \rangle \\ \vdots \\ \langle \phi_0 | f(A) | \phi_{l,0} \rangle \end{pmatrix}. \tag{A.9}$$

That is, we obtain $x_j = \langle \phi_0 | f(A) | j \rangle = \frac{1}{\sqrt{\kappa_l}} \langle \phi_0 | f(A) | \phi_{l,0} \rangle$ for all $j \in \Gamma_l(\alpha)$. □

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